PROPERTIES OF SYMMETRIC TENSORS OF VALENCE TWO AND THE STRUCTURE OF THE ELASTIC POTENTIAL

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Expressions for the components of a symmetric tensor of valence two defining its principal axes (1.8), (1.9) are given in terms of its trigonometric invariants and Euler's angles. Formulas are cited which use the physical components of the tensor given in the orthogonal coordinate system to define the invariants and the directions of the principal axes of the tensor. The conditions (2.4) that two tensors of the form under consideration are coaxial, are given.

In contrast to the usual approach [1], the elastic potential is considered here as a function of the trigonometric invariants of the strain tensor and of the Euler angles defining the directions of the principal axes of strain. Elementary considerations made it possible to establish the form of the functional dependence of the elastic potential on the Euler angles compatible with isotropy groups of the crystal classes and orientations. The cubic system however could not be dealt with by these means, since its generating symmetry elements include a third order axis inclined equally to the coordinate axes.

1. Let e_1, e_2, e_3 be the unit vectors of the principal vector basis of the symmetric tensor A, of valence two, i.e. the unit vectors directed along the principal directions of A.

Constructing the trigonometric tensor basis [2]

$$G_1 = {}^1/_3 \sqrt{3}(e_1e_1 + e_2e_2 + e_3e_3), \quad G_4 = {}^1/_2 \sqrt{2}(e_1e_3 + e_3e_1)$$
$$G_2 = {}^1/_6 \sqrt{6}(2e_2e_2 - e_1e_1 - e_3e_3), \quad G_5 = {}^1/_2 \sqrt{2}(e_1e_2 + e_2e_1)$$
$$G_3 = {}^1/_2 \sqrt{2}(e_1e_1 - e_3e_3), \qquad G_6 = {}^1/_2 \sqrt{2}(e_2e_3 + e_3e_2) \qquad (1.1)$$

we find that the tensor A has the following expansion:

$$\mathrm{A} = a_\alpha{}^\circ G_\alpha \qquad (\alpha = 1, \ldots, 6) \qquad (1.2)$$

with the coefficients given by

$$a_1{}^\circ = {}^1/_3 \sqrt{3} A_1, \quad a_2{}^\circ = \vartheta_a \sin\psi_a, \quad a_3{}^\circ = \vartheta_a \cos\psi_a, \quad a_4{}^\circ = a_5{}^\circ = a_6{}^\circ = 0 \qquad (1.3)$$
$$A_1 = a_{(1)} + a_{(2)} + a_{(3)}.$$

where A_1 is the first invariant,

$$\vartheta_a = {}^1/_3 \sqrt{3} \sqrt{(a_{(1)} - a_{(2)})^2 + (a_{(2)} - a_{(3)})^2 + (a_{(3)} - a_{(1)})^2}$$

ϑ_a denotes the intensity and ψ_a is the angle associated with the form of the tensor. The last three quantities are connected with the principal values $a_{(i)}$ of the tensor by the following relations:

$$a_{(1)} = {}^1/_3 A_1 - {}^1/_6 \sqrt{6}\, \vartheta_a \sin\psi_a + {}^1/_2 \sqrt{2}\, \vartheta_a \cos\psi_a$$

$$a_{(2)} = {}^1/_3 A_1 + {}^1/_3 \sqrt{6}\,\vartheta_a \sin\psi_a$$
$$a_{(3)} = {}^1/_3 A_1 - {}^1/_6 \sqrt{6}\,\vartheta_a \sin\psi_a - {}^1/_2 \sqrt{2}\,\vartheta_a \cos\psi_a$$

Let e_1', e_2', e_3' be an orthonormalized vector basis the passage to which from the principal vector basis is determined using the relations

$$e_i' = e_\alpha l_{\alpha i} \tag{1.4}$$

where l_{ki} are the coefficients of the matrix [3]

$$L = \| l_{ki} \| = \cos\omega \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + \frac{\sin\omega}{\omega} \begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix} + \\ + \frac{1-\cos\omega}{\omega^2} \begin{Vmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{Vmatrix} \tag{1.5}$$

$$\begin{aligned} &l_{11} = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta, \qquad l_{12} = -\cos\alpha\sin\gamma \\ &l_{13} = \cos\alpha\sin\beta\cos\gamma + \sin\alpha\cos\beta, \qquad l_{21} = \cos\beta\sin\gamma, \quad l_{22} = \cos\gamma \\ &l_{23} = \sin\beta\sin\gamma, \qquad l_{31} = -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta \\ &l_{33} = \sin\alpha\sin\gamma, \qquad l_{32} = -\sin\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta \end{aligned} \tag{1.6}$$

Here ω_1, ω_2, ω_3 are the components of the rotation vector transforming the unit vectors e_1, e_2, e_3 into e_1', e_2', e_3' and α, β, γ are the corresponding Euler angles (Fig. 1).

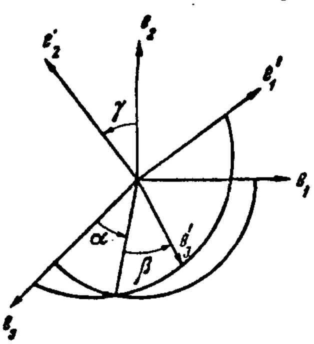

Fig. 1.

It can easily be shown that the basis G_1', . . ., G_6', constructed on e_1', e_2', e_3', admits the following expansion of the tensor A

$$A = a_\alpha G_\alpha' \tag{1.7}$$

with the coefficients

$$a_1 = {}^1/_3 \sqrt{3} A_1, \qquad a_j = \vartheta_a \sin\psi_a m_{2j} + \vartheta_a \cos\psi_a m_{3j} \tag{1.8}$$

where

$$\begin{aligned} &m_{22} = {}^1/_2 \cos^2\gamma - {}^1/_2, \qquad m_{32} = {}^1/_2 \sqrt{3} \cos 2\alpha \sin^2\gamma \\ &m_{23} = {}^1/_2 \sqrt{3}\cos 2\beta \sin^2\gamma, \qquad m_{33} = {}^1/_2 \cos 2\alpha \cos 2\beta (1 + \cos^2\gamma) - \sin 2\beta \sin 2\alpha \cos\gamma \\ &m_{24} = {}^1/_2 \sqrt{3}\sin 2\beta \sin^2\gamma, \qquad m_{34} = {}^1/_2 \sin 2\beta \cos 2\alpha (1 + \cos^2\gamma) + \cos 2\beta \sin 2\alpha \cos\gamma \\ &m_{25} = {}^1/_2 \sqrt{3}\cos\beta \sin 2\gamma, \qquad m_{35} = -{}^1/_2 \cos\beta \cos 2\alpha \sin 2\gamma + \sin\beta \sin 2\alpha \sin\gamma \\ &m_{26} = {}^1/_2 \sqrt{3}\sin\beta \sin 2\gamma, \qquad m_{36} = -{}^1/_2 \sin\beta \cos 2\alpha \sin 2\gamma - \cos\beta \sin 2\alpha \sin\gamma \end{aligned} \tag{1.9}$$

Here the following relations connect a_i with the physical components a_{ij} of the tensor A (on the e_1', e_2', e_3'- basis)

$$\begin{aligned} &a_1 = {}^1/_3 \sqrt{3}(a_{11} + a_{22} + a_{33}), \qquad a_4 = \sqrt{2}\, a_{13} \\ &a_2 = {}^1/_6 \sqrt{6}(2a_{22} - a_{11} - a_{33}), \qquad a_5 = \sqrt{2}\, a_{12} \\ &a_3 = {}^1/_2 \sqrt{2}(a_{11} - a_{33}), \qquad a_6 = \sqrt{2}\, a_{23} \end{aligned} \tag{1.10}$$

Performing the convolution of the symmetric tensors (contraction followed by symmetrization) [4]

$$A * B \equiv {}^1/_2 (AB + BA)$$

we obtain, in accordance with (1.1),

$$G_1 * G_1 = 1/3\sqrt{3}\,G_1,\quad G_1 * G_2 = 1/3\sqrt{3}\,G_2,\quad G_1 * G_3 = 1/3\sqrt{3}\,G_3 \tag{1.11}$$
$$G_2 * G_2 = 1/3\sqrt{3}\,G_1 + 1/6\sqrt{6}\,G_2,\quad G_2 * G_3 = -1/6\sqrt{6}\,G_3,\quad G_3 * G_3 = 1/3\sqrt{3}\,G_1 - 1/6\sqrt{6}\,G_2$$

2. The formulas (1.7) - (1.9) reveal the structure of the tensor. Knowing the invariants of the tensor and the directions of its pnncipal axes, we can compute its components on any orthonormalized basis. The converse problem is somewhat more involved. It can be stated as follows: given the physical components of the tensor in some orthogonal coordinate system, to find its principal values and the directions of its principal axes. First we find the invariants using the formulas

$$\vartheta_a = \sqrt{2/3 A_1^2 - 2A_2},\qquad \sin 3\psi_a = -3\sqrt{6}\,\vartheta_a^{-3}(A_3 - 1/3 A_1 A_2 + 2/27 A_1^3)$$
$$A_1 = a_{11} + a_{22} + a_{33}$$
$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} \qquad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

To determine the Euler's angles we must first compute A^2. Using the tables (1.11) we contract the tensor A with itself and denoting by $(a^2)_j$ the coefficients of the expansion of A^2 on the basis $G_1', \ldots, G_6'$ we find

$$\begin{aligned}
(a^2)_1 &= 1/6\sqrt{6}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) = 1/9\sqrt{3}A_1^2 + 1/3\sqrt{3}\vartheta_a^2 \\
(a^2)_2 &= 2/3 A_1 a_2 + 1/6\sqrt{6}(a_2^2 - a_3^2) + 1/12\sqrt{6}(a_5^2 + a_6^2 - 2a_4^2) \\
(a^2)_3 &= 2/3 A_1 a_3 - 1/3\sqrt{6}\,a_2 a_3 + 1/4\sqrt{2}(a_5^2 - a_6^2) \\
(a^2)_4 &= 2/3 A_1 a_4 - 1/3\sqrt{6}\,a_2 a_4 + 1/4\sqrt{2}(a_5^2 - a_6^2) \\
(a^2)_5 &= 2/3 A_1 a_5 + 1/6\sqrt{6}\,a_2 a_5 + 1/2\sqrt{2}\,a_3 a_5 + 1/2\sqrt{2}a_4 a_6 \\
(a^2)_6 &= 2/3 A_1 a_6 + 1/6\sqrt{6}\,a_2 a_6 - 1/2\sqrt{2}\,a_3 a_6 + 1/2\sqrt{2}a_4 a_5
\end{aligned} \tag{2.1}$$

On the other hand, from (1.2), (1.3) and (1.11) we have

$$A^2 = (1/9\sqrt{3}A_1^2 + 1/3\sqrt{3}\vartheta_a^2)\,G_1 + 2/3\,\vartheta_a(\sin\psi_a\,G_2 + 2/3 A_1 \cos\psi_a\,G_3) -$$
$$- 1/9\sqrt{6}\,\vartheta_a^2(\cos 2\psi_a\,G_2 + \sin 2\psi_a\,G_3)$$

Repeating the procedure used in deriving (1.8) we obtain

$$1/9\sqrt{3}A_1^2 + 1/3\sqrt{3}\vartheta_a^2 = (a^2)_1 \tag{2.2}$$
$$\sqrt{6}\,[(a^2)_j - 2/3 A_1 a_j] = -\vartheta_a^2(\cos 2\psi_a m_{2j} + \sin 2\psi_a m_{3j}) \qquad (j = 2, 3, \ldots, 6)$$

Solving the pairs of equations constructed for each value of j from (1.8) and (2.2), we obtain

$$m_{2j} = \left\{ -\frac{\sqrt{6}\cos\psi_a}{\vartheta_a^2 \cos 3\psi_a}[(a^2)_j - 2/3 A_1 a_j] - \frac{\sin 2\psi_a}{\vartheta_a \cos 3\psi_a} a_j \right\} \tag{2.3}$$
$$m_{3j} = \left\{ \frac{\sqrt{6}\sin\psi_a}{\vartheta_a^2 \cos 3\psi_a}[(a^2)_j - 2/3 A_1 a_j] + \frac{\cos 2\psi_a}{\vartheta_a \cos 3\psi_a} a_j \right\} \quad (j = 2, 3, \ldots, 6)$$

Using the formulas obtained we can now solve the converse problem. Thus using (1.10) we obtain a_k from the known physical components a_{ij} of the tensor. These in turn are used to obtain m_{2j} and m_{3j} from (2.1) and (2.3). Then the (simultaneous) system of equations (1.9) yields the Euler angles defining the position of the principal axes of the tensor A.

It is easy to see that the conditions that two symmetric tensors A and C are coaxial, are

$$m_{2j}^{c} = m_{2j}^{a}, \quad m_{3j}^{c} = m_{3j}^{a} \tag{2.4}$$

The latter, together with (2.3), yield the following relations connecting the coaxial tensors

$$(S_c = (C - {}^1/_3 C_1 G)/\vartheta_c, \quad S_a = (A - {}^1/_3 A_1 G)/\vartheta_a$$

$$S_c = \frac{\cos(2\psi_a + \psi_c)}{\cos 3\psi_a} S_a + \sqrt{6}\,\frac{\sin(\psi_a - \psi_c)}{\cos 3\psi_a}(S_a^2 - {}^1/_3 G)$$

$$S_c^2 - {}^1/_3 G = \frac{\cos(\psi_a + 2\psi_c)}{\cos 3\psi_a}(S_a^2 - {}^1/_3 G) + \frac{\sin 2(\psi_a - \psi_c)}{\sqrt{6}\cos 3\psi_a} S_a$$

The first of the above conditions was obtained in [5]. Condition (2.4) represents a tensor analog of the known condition of parallelism of two vectors c and a

$$c_1/a_1 = c_2/a_2 = c_3/a_3$$

The Euler angles can also be obtained, one after another, from the following formulas:

$$\operatorname{tg} 2\beta = \frac{2(a_2 - \vartheta_a \sin\psi_a) a_4 - \sqrt{3}\, a_5 a_6}{2(a_2 - \vartheta_a \sin\psi_a) a_3 + \sqrt{3}(a_6^2 - a_5^2)} \qquad (0 \leqslant \beta \leqslant 2\pi)$$

$$\operatorname{ctg}\gamma = \frac{\cos 2\beta a_4 - \sin 2\beta a_3}{\sin\beta a_5 - \cos\beta a_6} \qquad (0 \leqslant \gamma \leqslant \pi)$$

$$\operatorname{tg} 2\alpha = -\frac{\sqrt{3}(a_5 \sin\beta - a_6 \cos\beta)}{\sin\gamma\,[a_2 + \sqrt{3}(\cos 2\beta a_3 + \sin 2\beta a_4) - \vartheta_a \sin\psi_a]} \qquad (0 \leqslant \alpha \leqslant 2\pi)$$

In the plane case (i.e. when $\gamma \equiv 0$) we have a single angle of rotation of the principal axes $\eta = \alpha + \beta$. Moreover, from (1.8) and (1.9) we have

$$\eta = {}^1/_2 \operatorname{arctg}(a_4/a_3) \qquad (0 \leqslant \eta \leqslant 2\pi)$$

3. Let us denote by $\Phi(e_{11}, \ldots, e_{23})$ the density of the elastic strain energy. The presence of the symmetry elements in the elastic material imposes definite restrictions on the form of the arguments of Φ. A detailed analysis of this problem is given in [1]. It should be noted however, that the combinations of the components of strain obtained in [1] as arguments of Φ are quite complicated and numerous.

Relations (1.8) - (1.10) imply that we may consider

$$\Phi = \Phi(E_1, \vartheta_e \sin\psi_e, \quad \vartheta_e \cos\psi_e; \quad \alpha, \beta, \gamma) \tag{3.1}$$

The main purpose of this paper is to show what restrictions on the form of the arguments of the given function are imposed by the presence of the elastic symmetry elements in the material.

Let us consider, in addition to the basis e_1', e_2', e_3' another orthonormalized basis e_1'', e_2'', e_3'', the passage to which can be realized by an additional rotation such that $e_j'' = e_\beta' l_{\beta j}'$. Relation (1.4) gives $e_j'' = e_\alpha l_{\alpha\beta} l_{\beta j}'$ i.e. $e_j'' = e_\alpha \lambda_{\alpha j}$, where λ_{ij} denote the components of the matrix

$$\Lambda = \|\lambda_{kl}\| = LL' \qquad (k, l = 1, 2, 3) \tag{3.2}$$

Thus the successive passage from the initial basis e_1, e_2, e_3 to the intermediate basis

e_1', e_2', e_3' (defined by the matrix L) and to the final basis e_1'', e_2'', e_3'' (the matrix L') can be replaced by a straight passage from the initial to the final basis by means of the matrix (3.2). Let for example $L' = L_{2'}^{2\pi/\omega'}$ be the matrix describing the rotation about the e_2'-axis through an angle ω'. We have $\omega_1' = \omega_3' = 0$ and $\omega_2' = \omega'$ and by (1.5)

$$L' = L_2^{2\pi/\omega'} = \begin{Vmatrix} \cos\omega' & 0 & \sin\omega' \\ 0 & 1 & 0 \\ -\sin\omega' & 0 & \cos\omega' \end{Vmatrix}$$

Right-multiplying this matrix by L (1.6) we obtain

$$\lambda_{11} = \cos\alpha\cos(\beta+\omega')\cos\gamma - \sin\alpha\sin(\beta+\omega'), \qquad \lambda_{12} = \cos\alpha\sin\gamma$$
$$\lambda_{13} = \cos\alpha\sin(\beta+\omega')\cos\gamma + \sin\alpha\cos(\beta+\omega'), \qquad \lambda_{21} = \cos(\beta+\omega')\cos\gamma$$
$$\lambda_{22} = \cos\gamma,\ \lambda_{23} = \sin(\beta+\omega')\sin\gamma, \quad \lambda_{31} = -\sin\alpha\cos(\beta+\omega')\cos\gamma - \cos\alpha\sin(\beta+\omega')$$
$$\lambda_{32} = \sin\alpha\sin\gamma, \qquad \lambda_{33} = -\sin\alpha\sin(\beta+\omega')\cos\gamma + \cos\alpha\cos(\beta+\omega')$$

which, compared with (1.6), shows that a straightforward rotation from the initial to the final basis is determined by the Euler angles

$$L_{2'}^{2\pi/\omega'} \quad \{\alpha' = \alpha,\ \beta' = \beta + \omega',\ \gamma' = \gamma\}$$

Similarly, an additional rotation about the e_3'-axis by π, yields two possible Euler's angles

$$L_{3'}^{2\pi/\omega'} = L_{3'}^{2} \quad \begin{cases} \text{a)}\ \alpha' = \pi + \alpha,\ \beta' = \pi - \beta,\ \gamma' = \pi - \gamma \\ \text{b)}\ \alpha' = \alpha, \qquad \beta' = -\beta, \quad \gamma' = \pi + \gamma \end{cases}$$

corresponding to the same angle of rotation cosines.

Let us dwell on the second variant. The expressions obtained yield

$$\begin{aligned}
&L_{2'}^{2} && \{\alpha' = \alpha,\ \beta' = \beta + \pi,\ \gamma' = \gamma\} \\
&L_{2'}^{3} && \{\alpha' = \alpha,\ \beta' = \beta + {}^2/_3\pi,\ \gamma' = \gamma\} \\
&L_{2'}^{4} && \{\alpha' = \alpha,\ \beta' = \beta + {}^1/_2\pi,\ \gamma' = \gamma\} \\
&L_{2'}^{6} && \{\alpha' = \alpha,\ \beta' = \beta + {}^1/_3\pi,\ \gamma' = \gamma\} \\
&L_{2'}^{\infty} && \{\alpha' = \alpha,\ \beta' = \beta + \omega',\ \gamma' = \gamma\} \quad (\omega' \text{ is arbitrary}) \\
&L_{3'}^{2} && \{\alpha' = \alpha,\ \beta' = -\beta,\ \gamma' = \gamma + \pi\}
\end{aligned} \tag{3.3}$$

4. The crystallographic classes (except the cubic class) and five orientations have the following elements generating the symmetry groups (g) given in the third column of Table 1 [6] where n, and $\bar{n}$ denote the n-th order proper and mirror axes, m is the plane of reflection, $(\cdot)$ denotes parallelism and $(:)$ denotes the perpendicularity of the symmetry elements. The last column gives the generating elements of the characteristic symmetry rotation subgroups. As we know [6, 7] the defining elements g can be obtained from g_0 by supplementing the latter with the inversion transformation i. Since the components of a tensor of valence two are invariant under the inversion transformation we can limit ourselves to considering the column g_0.

Let us e.g. consider the rhombic system whose generating elements are, in accordance with Table 1, $L_{2'}^{2}$ and $L_{3'}^{2}$. By the first and sixth equation of (3.3) the Euler's angles must appear in the elastic potential (3.1) in the form invariant under the transformation

$$\alpha' = \alpha, \quad \gamma' = \gamma + \pi, \quad \beta' = \beta + \pi, \quad \beta' = -\beta$$

Since we only consider here the 2π-periodic functions of the Euler angles, the following functional arguments satisfy the above condition: $\sin\alpha$, $\cos\alpha$, $\cos 2\beta$, $\sin 2\gamma$, $\cos 2\gamma$

Table 1.

Crystallographic Systems	Class No. (acc. to Groth)	Generating Elements g	g_0
Triclinic	1	1	1
	2	$\bar{2}$	1
Monoclinic	3	2	2
	4	m	2
	5	$2:m$	2
Rhombic	6	$2:2$	$2:2$
	7	$2\times m$	$2:2$
	8	$m\times 2:m$	$2:2$
Tetragonal	9	$\bar{4}$	4
	10	4	4
	11	$4\times m$	$4:2$
	12	$4:2$	$4:2$
	13	$4:m$	4
	14	$4\times m$	$4:2$
	15	$m\times 4:m$	$4:2$
Trigonal	16	3	3
	17	$\bar{6}$	3
	18	$3:2$	$3:2$
	19	$3\times m$	$3:2$
	20	$\bar{6}\times m$	$3:2$
Hexagonal	21	$3:m$	6
	22	$m\times 3:m$	$6:2$
	23	6	6
	24	$6:2$	$6:2$
	25	$6:m$	6
	26	$6\times m$	$6:2$
	27	$m\times 6:m$	$6:2$
Orientations		∞	∞
		$\infty:m$	∞
		$\infty\times m$	$\infty:2$
	(gyrotropic)	$m\times\infty:m$	$\infty:2$
	(isotropic)	$\infty:2$	$\infty:2$

The remaining crystallographic systems and orientations are treated in a similar manner, and the results obtained are

Triclinic (system)

$$\Phi\{{}^1/_3\sqrt{3}E_1,\ \vartheta_e\sin\psi_e,\ \vartheta_e\cos\psi_e;\ \sin\alpha,\ \cos\alpha,\ \sin\beta,\ \cos\beta,\ \sin\gamma,\ \cos\gamma\}$$

Monoclinic

$$\Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \sin 2\beta,\ \cos 2\beta,\ \sin\gamma,\ \cos\gamma\}$$

Rhombic

$$\Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \cos 2\beta,\ \sin 2\gamma,\ \cos 2\gamma\}$$

Tetragonal (classes)

9, 10, 13 $\quad \Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \sin 4\beta,\ \cos 4\beta,\ \sin\gamma,\ \cos\gamma\}$

11, 12, 14, 15 $\quad \Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \cos 4\beta,\ \cos 2\gamma,\ \sin 2\gamma\}$

Trigonal (4.1)

16, 17 $\quad \Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \sin 3\beta,\ \cos 3\beta,\ \sin\gamma,\ \cos\gamma\}$

18, 19, 20 $\quad \Phi\{\ldots,\ \sin\alpha,\ \cos\alpha,\ \cos 3\beta,\ \sin 2\gamma,\ \cos 2\gamma\}$

Hexagonal	21, 23, 25	$\Phi\{\ldots, \sin\alpha, \cos\alpha, \sin 6\beta, \cos 6\beta, \sin\gamma, \cos\gamma\}$
	22, 24, 26, 27	$\Phi\{\ldots, \sin\alpha, \cos\alpha, \cos 6\beta, \sin 2\gamma \cos 2\gamma\}$
Orientations	∞, $\infty : m$	$\Phi\{\ldots, \sin\alpha, \cos\alpha, \sin\gamma, \cos\gamma\}$
	$\infty \times m$, $\infty : 2$, $m \times \infty : m$	$\Phi\{\ldots, \sin\alpha, \cos\alpha, \sin 2\gamma, \cos 2\gamma\}$

Here the dots denote the invariant arguments $^1/_3\sqrt{3}\ E_1$, $\vartheta_e \sin\psi_e$, and $\vartheta_e \cos\psi_e$ which have been omitted. They can also be replaced by $e_{(1)}$, $e_{(2)}$ and $e_{(3)}$.

5. Let us consider the orientations ∞ and $\infty : m$, normally defining a transversally isotropic material. Investigation of the polynomial functions ([1] p. 36) and of the general type functions [8] of the strain components yields the following arguments written in the notation adopted here:

$$I_e, \quad II_e, \quad III_e, \quad e_{22}, \quad e_{12}^2 + e_{23}^2$$

Formulas (1.8) - (1.10) yield

$$e_{22} = {}^1/_3 E_1 + {}^1/_3 \sqrt{6}\{\vartheta_e \sin\psi_e ({}^3/_2 \cos^2\gamma - {}^1/_2) + {}^1/_2 \sqrt{3}\,\vartheta_e \cos\psi_e \cos 2\alpha \sin^2\gamma\}$$

$$e_{12}^2 + e_{23}^2 = (\vartheta_e \sin\psi_e)^2\{{}^3/_8 \sin^2 2\gamma\} + (\vartheta_e \sin\psi_e)(\vartheta_e \cos\psi_e)\{-{}^1/_4 \sqrt{3}\cos 2\alpha \sin^2 2\gamma\} + (\vartheta_e \cos\psi_e)^2\{{}^1/_8 \cos^2 2\alpha \sin^2 2\gamma + {}^1/_2 \sin^2 2\alpha \sin^2\gamma\}$$

Thus we see that the usual approach yields the Euler angles for the transversally isotropic material, which can appear as the arguments of Φ only within the two combinations given above. With the approach employed in the present paper, we see from (4.1) that any periodic functions of α and γ can be used as the arguments of Φ. This of course also applies to the remaining systems and orientations considered here.

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